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BI-IDEALS AND QUASI-IDEALS OF BCK-ALGEBRA

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ABSTRACT

In this paper, we introduce the concept of Bi-ideals and Qusai-ideals of BCK-algebra. Some of its effects with examples were also given.

KEYWORDS: Bck-algebra, Bi-ideal, Qusai-ideal.

1. INTRODUCTION

In 1966, Y. Imai and K. Iseki [3] introduced a new notation, called BCK-algebra.

This notion is originated from two different ways: One of them is based on set theory; another is from classical and non-classical propositional calculi. As is well known, there is a close relationship between the notions of the set difference in set theory and the implication functor in logical systems. Y. B. Jun [1] deal with various results on ideals of BCK algebras. The impression of bi-ideal for semi groups was interrupted by Good and Hughes. T. Tamizh Chelvam et al.[2] innovated certain concepts on bi-ideals of near rings. I. Yakabe [4] constituted several properties on Qusai ideals in near rings.

2. PRELIMINARIES

In this section, we reproduce some basic definitions which are essential for the development of the paper.

Definition: 2.1

Let X be a set with a binary operation $*$ and a constant 0. Then $(X,*,0)$ of type (2, 0) is called a **BCK-algebra** if it satisfies the following conditions:

i. $((x * y) * (x * z)) * (z * y) = 0$

ii. $(x * (x * y)) * y = 0$

iii. $x * x = 0$

iv. $0 * x = 0$

v. $x * y = 0$ and $y * x = 0 \Rightarrow x = y \ \forall x, y \in X$.

We can define a partial ordering " $\leq$ " on X by $x \leq y$ if and only if $x * y = 0$. In any BCK-algebra X, the following hold:

i. $x * 0 = 0$

ii. $x * y \leq x$

iii. $(x * y) * z = (x * z) * y$

iv. $(x * z) * (y * z) \leq x * y$

v. $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

Example: 2.2
Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following cayley table:

$*$	**0**	***a***	***b***	***c***	***d***
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	a	c	0	a
d	d	d	d	d	0

Definition: 2.3
A BCK-algebra X is said to be **Positive implicative** if $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$.

Definition: 2.4
A non-empty subset S of a BCK-algebra X is called a **BCK-Subalgebra** of X if $x * y \in S$ whenever $x, y \in S$.

Definition: 2.5
A non-empty subset I of a BCK-algebra X is called an **Ideal** of X if i. $0 \in I$
ii. $x * y \in I$ and $y \in I$ imply $x \in I$.

For any $a \in X$ let (a) denote the set of all elements of X which are less than or equal to a, i.e., $(a) = \{x \in X | x \leq a\}$. Note that $0 \in (a)$, and (a) is not an ideal of X.

Definition: 2.6
Let A and B be two non-empty subsets of X. We shall define two types of products:
$AB = \{ \sum a_i b_i | a_i \in A, b_i \in B\}$ and $A * B = \{\sum(a_i(a' + b_i) - a_i a_i') | a_i, a_i' \in A, b_i \in B\}$,
Where $\sum$, denotes all possible additions of finite terms. In case that $B = \{b\}$, we denote AB by Ab.

Definition: 2.7
A subgroup S of X is called an **X-subgroup** of X if $XS \subseteq S$.

Definition: 2.8
An element $a \in X$ is called an **idempotent** if $a^2 = a$.

Definition: 2.9
A subgroup M of a BCK-algebra X is called a **BCK-subalgebra** if $MM \subseteq M$.

Defintion: 2.10
An element a in a BCK-algebra X is said to be **Regular** if $a \in aXa$. A BCK-algebra X is said to be **Regular** if every element in X is regular, i.e., for every $a \in X$, there exists a

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$b \in X$ such that $a = aba$.

Defintion: 2.11
Let A be a set. $M \subseteq (A)$ (where (A) denote the power set of A) is said to be a

Moore-system on A ⟺

i. $A \subseteq M$.
ii. For any set I, $(\forall i \in I: M_i \in M) \Rightarrow \bigcap_{i \in I} M_i \in M$.

3. BI-IDEAL OF BCK-ALGEBRA:

In this section, we introduced the concept of Bi-ideal of BCK-algebra and discuss some of its effects.

Definition: 3.1
Let B is a subalgebra of X and $BXB \cap (BX) * B \subseteq B$ is called a **Bi-ideal** of BCK- algebra. In case of zero symmetric $BXB \subseteq B$.

Example: 3.2
Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following cayley table:

$*$	**0**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
0	0	0	0	0	0
$\boldsymbol{a}$	a	0	0	0	0
$\boldsymbol{b}$	b	a	0	a	0
$\boldsymbol{c}$	c	c	c	0	0
$\boldsymbol{d}$	d	d	d	d	0

Clearly, $B = \{0, a, b\}$ is a bi-ideal of BCK-algebra X.

Proposition: 3.3
The set of all bi-ideals of a BCK-algebra X form a Moore system on X.
Proof:
Let $\{B_i\}_{\in I}$ be a set of bi-ideals in X. Let $B = \bigcap_{i \in I} B_i$
Then $BXB \cap (BX) * B \subseteq B_iXB_i \cap (B_iX) * B_i \subseteq B_i$ for every $i \in I$. Therefore B is a bi-ideal of X.

Proposition: 3.4
If B be a bi-ideal of a BCK-algebra X and S is a BCK-subalgebra of X, then $B \cap S$ is a bi-ideal of S.
Proof:
Since B is a bi-ideal of X, $BXB \cap (BX) * B \subseteq B$. Let $C = B \cap S$. Now $CSC \cap (CS) * C =$
$(B \cap S)(B \cap S) \cap ((B \cap S)S) * (B \cap S)$

$$\subseteq BSB \cap S \cap (BS) * B$$

$$\subseteq B \cap S = C$$

$CSC \cap (CS) * C \subseteq C$. Hence C is a bi-ideal of S.

Hence $B \cap S$ is a bi-ideal of S.

Proposition: 3.5

Let X be a zero-symmetric BCK-algebra. A subalgebra B of X is a bi-ideal if and only if $BXB \subseteq B$.

Proof:

For a subalgebra B of $(X,* ,0)$, if $BXB \subseteq B$, then B is a bi-ideal of X.

Conversely,

If B is a bi-ideal, we have $BXB \cap (BX) * B \subseteq B$. To prove $BXB \subseteq B$.

Since X is a zero-symmetric, $XB \subseteq X * B$. We get $BXB = BXB \cap BXB$

$\subseteq BXB \cap (BX) * B$

$\subseteq$

B i.e., $BXB \subseteq B$.

Proposition: 3.6

Let X be a zero-symmetric BCK-algebra. If B is a bi-ideal of X, then Bx and $x'B$ are bi-ideals of X where $x, x \in X$ and x' is distributive element in X.

Proof:

Clearly, Bx is a subalgebra of $(X,* ,0)$ and $Bx\ X\ Bx \subseteq B\ X\ Bx \subseteq Bx$.

We get Bx is a bi-ideal of X.

Again $x'B$ is a subalgebra. Since x' is distributive in X and

$x'B\ X\ x'B \subseteq x'B\ X\ B \subseteq x'B$. Thus $x'B$ is a bi-ideal of X.

Therefore Bx and $x'B$ are bi-ideals of X.

Corollary: 3.7

If B is a bi-ideal of a zero-symmetric BCK-algebra X and b is a distributive element in X, then bBc is a bi-ideal of X, where $c \in X$.

Proof:

Given b is a distributive element in X, we get bB is a bi-ideal of X.

Clearly, bBc is a subalgebra of $(X,* ,0)$. To prove, bBc is a bi-ideal of X.

Now $bBc\ X\ bBc \subseteq B\ X\ bBc$

$\subseteq bBc$

Thus bBc is a bi-ideal of X.

4. QUSAI-IDEAL OF BCK-ALGEBRA:

In this section, we introduced the concept of Qusai-ideal of BCK-algebra and discuss some of its effects.

Definition: 4.1

A subalgebra Q of a BCK-algebra X is called a **qusai-left (qusai-right) ideal** of X if i. $0 \in Q$

ii. $x \in Q, y \in Q \Rightarrow y \wedge x \in Q$ $(x \wedge y \in Q)$

Q is called a **qusai-ideal** if it satisfies both qusai-left and qusai-right ideal. In case of zero symmetric, $QX \cap XQ \subseteq Q$.

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Example: 4.2
Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following cayley table:

$*$	**0**	***a***	***b***	***c***	***d***
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	c	c	0	0
d	d	d	d	d	0

Clearly, $A = \{0, a, b, c\}$ be the qusai-ideal of a BCK-algebra X.

Proposition: 4.3
The set of all qusai-ideals of a BCK-algebra X forms a Moore-system on X.
Proof:
Let $(\lambda \in \Lambda)$ be any set of qusai-ideals of X.
Then $\bigcap_{\lambda\in\Lambda} Q_\lambda$ is clearly a subalgebra of $(X, *, 0)$. Moreover, for every $(\mu \in \Lambda)$.

We have $D = (\bigcap_{\lambda\in\Lambda} Q_\lambda)\, X \cap X\, (\bigcap_{\lambda\in\Lambda} Q_\lambda) \cap X * (\bigcap_{\lambda\in\Lambda} Q_\lambda) \quad (\bigcap_{\lambda\in\Lambda} Q_\lambda \subseteq Q_\mu)$

$\subseteq Q_\mu\, X \cap X\, Q_\mu \cap X * Q_\mu$

$\subseteq Q_\mu$

Hence $D \subseteq \bigcap_{\lambda\in\Lambda} Q_\lambda$, that is, $\bigcap_{\lambda\in\Lambda} Q_\lambda$ is a qusai-ideal of X.

Proposition: 4.4
The intersection of a qusai-ideal Q and a BCK-subalgebra M of a BCK-algebra X is a qusai-ideal of M.
Proof:
Clearly, $Q{\cap}M$ is a subalgebra of $(M, +)$.

Moreover, we have $(Q{\cap}M){\cap}M(Q{\cap}M){\cap}M * (Q{\cap}M)$

$\subseteq (Q{\cap}M)M{\cap}M(Q{\cap}M)$
$\subseteq MM \subseteq M$

and $(Q{\cap}M)M{\cap}M(Q{\cap}M){\cap}M * (Q{\cap}M)$

$\subseteq QX{\cap}XQ{\cap}X * Q$

$\subseteq Q$

These imply that $Q{\cap}M$ is a qusai-ideal of M.

Proposition: 4.5
Let X be a zero-symmetric BCK-algebra. Then a subalgebra Q of $(X, *, 0)$ is a qusai- ideal of X if and only if $QX{\cap}X\,Q \subseteq Q$.
Proof:
We first remark that $X\,Q \subseteq X * Q$. In fact, for any elements x of X and q of Q, we have $xq = (0 + q) - x0$.
Since X is zero-symmetric. Hence $X\,Q \subseteq X * Q$.

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From this property, we have $QX \cap X\ Q \cap X * Q = QX \cap XQ$, by which this proposition is easily seen.

5. BI-IDEALS WHICH ARE ALSO QUSAI-IDEALS:

In this section, we discuss the relationship between a bi-ideals and a qusai-ideals in BCK-algebra.

Proposition: 5.1
Let B be a bi-ideal of a BCK-algebra X. If B is itself a regular BCK-algebra, then any bi-ideal of B is a bi-ideal of X.
Proof:

Let A be a bi-ideal of B.

Since B is regular, for $a \in A \subseteq B$, $a = aba$ for some $b \in B$ and so $A \subseteq AB \cap BA$.

Thus $AXA \subseteq (AB)(BA)$

$\subseteq A(BXB)A$

$\subseteq ABA$

$AXA \subseteq A$

i.e., A is a bi-ideal of X.

Proposition: 5.2
Let X be a BCK-algebra and B a bi-ideal of X. If elements of B are regular, then B is a qusai-ideal of X.
Proof:

Let $x \in BX \cap XB$.

Then $x = bn = n'b'$ for some $b, b' \in B$ and $n, n' \in X$.

Since B is regular, $b = bb_1b$ for some $b_1 \in B$.
Hence $x = bn = (bb_1b)$
$= (bb_1)(bn)$
$= bb_1n'b' \in BXB \subseteq B$
i.e., $BX \cap XB \subseteq B$

Hence B is a qusai-ideal of X.

Corollary: 5.3
If B is a bi-ideal and a regular BCK-subalgebra of X, then any bi-ideal of B is a qusai- ideal of X as well as of B. If Q is a qusai-ideal of X which is itself regular, then any qusai- ideal of Q is also a qusai-ideal of X.
Proof:

Let B is a bi-ideal of X. Let A be a bi-ideal of B.

Since B is regular subalgebra of X.

To prove A is a qusai-ideal of X.

i.e., To prove $AX \cap XA \subseteq A$.

Let $a \in A \subseteq B$, $a = aba$ for some $b \in B$.

So $A \subseteq AB \cap BA \Rightarrow A \subseteq AB$ and $A \subseteq BA$.

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Thus $AX \cap XA \subseteq (AB) \cap X(AB)$

$\subseteq AB(X \cap X)BA$

$\subseteq A(BXB)A$

$\subseteq ABA \subseteq A$

Hence A is a qusai-ideal of X.

To prove A is a qusai-ideal of B.

i.e., To prove $AB \cap BA \subseteq A$. Already we know, A is a bi-ideal of B.

$AB \cap BA = A(B \cap B)A$

$= ABA \subseteq A$

$AB \cap BA \subseteq A$.

Given $QX \cap XQ \subseteq Q$ and Q is regular. Let Q' be a qusai-ideal of Q.

To prove Q' is a qusai-ideal of X.

i.e., To prove $Q'X \cap XQ' \subseteq Q'$.

Let $q' \in Q' \subseteq Q \Rightarrow q' = q'qq'$ for some $q \in Q$

So $Q' \subseteq QQ' \cap Q'Q \Rightarrow Q' \subseteq QQ'$ and $Q' \subseteq Q'Q$

Thus $Q'X \cap XQ' \subseteq (Q'Q) \cap X(QQ')$

$= Q'(QX \cap XQ)Q'$

$\subseteq Q'QQ'$

$Q'X \cap XQ' \subseteq Q'$

Hence Q' is a qusai-ideal of X.

Corollary: 5.4
A subalgebra M of a regular BCK-algebra is a qusai-ideal if and only if M is a bi-ideal of X.
Proof:

Assume that M is a qusai-ideal of X. To prove M is a bi-ideal of X.

Now $MXM \cap (MX) * M \subseteq M(X \cap X)M \cap (MX) * M$

$\subseteq MX \cap XM \cap X * M \subseteq M$

Hence M is a bi-ideal of X.

Conversely,

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Assume that M is a bi-ideal of X. To prove M is a qusai-ideal of X.

Now $MX \cap XM \cap X * M \subseteq (X \cap X) \cap (MX) * M$

$\subseteq MXM \cap (MX) * M \subseteq M$

Hence M is a qusai-ideal of X.

Corollary: 5.5

A subalgebra M of a regular BCK-algebra X is a qusai-ideal of X if and only if M satisfies the condition $MXM \subseteq M$.

Proof:

Assume that M is a qusai-ideal of X. To prove the condition $MXM \subseteq M$.

Now $MXM \subseteq (X \cap X)$

$\subseteq MX \cap XM \subseteq M$

Hence the condition $MXM \subseteq M$ is proved.

Conversely,

Assume that the condition $MXM \subseteq M$ is true. To prove M is a qusai-ideal of X.

Now $MX \cap XM \subseteq (X \cap X)$

$\subseteq MXM \subseteq M$

Hence M is a qusai-ideal of X.

Proposition: 5.6

Let X be a regular BCK-algebra in which idempotents commute. Then every qusai- ideal of X is idempotent.

Proof:

Let M be qusai-ideal of X and $a \in M$. Since is a BCK-subalgebra, $M^2 \subseteq M$ and so we have only to prove that $M \subseteq M^2$. i.e., $a \in M^2$.

By the regularity of X we have $a = axa$.

Here xa is an idempotent and xa is in the center of X by [7] Theorem1. Using $MX^2M \subseteq MX \cap XM \subseteq M$

We get $a = (ax)(xa) = (ax)(xa)a$

$= (ax^2a) \in (MX^2M) \subseteq M^2$

Proposition: 5.7

Let X be a BCK-algebra in which every qusai-ideal is idempotent. Then, for left X- subalgebra L and right X-subalgebra R of X, $RL = R \cap L \subseteq LR$ is true.

Proof:

Let A and B are two qusai-ideals in X, then $A \cap B$ is also a qusai-ideal. By the idempotence of $A \cap B$ we have $A \cap B = (A \cap B)^2 \subseteq AB \cap BA$.

On the otherhand $AB \cap BA \subseteq AX \cap XA \subseteq A$.

Similarly $AB \cap BA \subseteq XB \cap BX \subseteq B$.

And so $A \cap B = AB \cap BA$.

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Now, let L be a left X-subalgebra and R be a right X-subalgebra of X. Since X- subalgebra are always qusai-ideals, we have $R \cap L = RL \cap LR$, but $RL \subseteq R \cap L$ and so $RL = R \cap L \subseteq LR$.

Proposition: 5.8

Let R and L be respectively right and left X- subalgebra of X. Then any subalgebra B of X such that $RL \subseteq B \subseteq R \cap L$ is a bi-ideal of X.

Proof:

For a subalgebra B of $(X, * ,0)$ with $RL \subseteq B \subseteq R \cap L$.

We have $BXB \subseteq (R \cap L)(R \cap L)$

$\subseteq RXL \subseteq RL \subseteq B$ and so B is a bi-ideal of X.

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